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ON SOME GENERAL VARIATIONAL PRINCIPLES FOR CREEP WITH APPLICATIONS TO THIN SHELLS

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Abstract—A constitutive relation for a viscous material subject to small strains but finite rotations is postulated and associated variational theorems are formulated. These are similar to the principle of minimum of potential energy and the Hellinger–Reissner theorem of an elastic solid. The derivation of strain–displacement relations for thin shells subject to small strains but moderately large rotations are given. On this basis a mixed variational principle for thin viscous shells is developed. For the problem of creep collapse of long cylindrical shells under external pressure it is demonstrated that the mixed variational principle may be advantageous compared to other variational theorems. A comparison with the creep collapse theory of Hoff *et al.* is given.

1. INTRODUCTION

Most engineering studies of creep in structural members apply either one of the two constitutive equations: In the first case it is assumed that the total strain rate is obtained by combining an elastic strain rate, a creep rate and a thermal strain rate in an additive manner. Here the creep rate may contain secondary and/or primary creep. This constitutive equation relates the total strain rate to the stress and the stress rate (and possibly other variables which describe the hardening of the material). The second constitutive relation relates the (total) strain rate to the stress (and possibly other variables describing the hardening of the material); an important example in this class of constitutive relations is the case of secondary creep according to Odqvist's invariant theory[1].

We will restrict attention to those problems where the elastic strain may be safely neglected, i.e. the second class of constitutive equations. These equations together with the strain–displacement relations, the quasistatic equilibrium equations, the boundary and initial conditions represent an initial-boundary value problem. For generality it will be assumed that these equations are formulated for small strains but finite rotations and material coordinates will be used. If one assumes that the displacements are given at some instant t, then the field equations and boundary conditions represent a boundary value problem for the stresses and displacement rates.

For infinitesimal displacement gradients variational theorems are well known which apply to this boundary value problem (see, e.g. Refs. [2, 3]). They are analogous to the principles of mimimum of potential or complementary energy. In this paper several variational theorems will be presented which are more general than those mentioned above. They are formulated for small strains but finite rotations and are similar to the principle of minimum of potential energy and the Hellinger–Reissner theorem in elasticity[4–6].

The condition of small strains but large rotations is sometimes typical for problems in thin plates or shells especially if creep collapse is involved. Therefore a mixed variational principle for thin shells is developed where stress resultants and displacement rates are to be varied independently. This shell theory is within the frame of the Kirchhoff–Love hypothesis and is restricted to small strains and a small rotation around the normal but to moderately large rotations of the normal. For the case of infinitesimal rotations this principle can be reduced to that given by Rabotnov[7] and Grigoliuk and Lipovtsev[8] for a cylindrical shell under axisymmetric loading.

Finally a relatively simple example is used to demonstrate the fact that the application of the mixed variational principle may lead in certain cases to a quasilinear system of differential equations. This is a definite advantage compared to the other variational theorems. The problem considered is the creep collapse of a cylindrical shell under external uniform pressure. It has been discussed by various authors, e.g. Hoff *et al.*[9]. A discussion of some of these theories can be found in Ref. [10]. In recent years this problem had some significance for the design of fuel element claddings in steam or gas cooled Fast Breeder Reactors.

The general theorems and analysis presented in this paper can be extended to a nonlinear viscoelastic material, i.e. the first constitutive relation[10, 11].

2. THE BOUNDARY VALUE PROBLEM

Since finite displacement gradients will be considered it is essential to distinguish between the material and spatial description of the flow of matter. In this study only material (Lagrangian) coordinates x^k ($k = 1, 2, 3$) will be used. Let u_k and u^k be co- and contravariant components of the displacement vector $\mathbf{u}$ with respect to a reference coordinate system in the undeformed body. Then the Lagrangian strain tensor and the strain rate tensor are defined as

$$e_{kl} = \tfrac{1}{2}(u_{k;l} + u_{l;k} + u^m{}_{;k} u_{m;l}) \tag{1}$$

$$\begin{aligned} \dot{e}_{kl} &= \frac{\partial}{\partial t} e_{kl}(x^i, t) \\ &= \tfrac{1}{2}(\dot{u}_{k;l} + \dot{u}_{l;k} + \dot{u}^m{}_{;k} u_{m;l} + u^m{}_{;k} \dot{u}_{m;l}). \end{aligned} \tag{2}$$

The usual summation convention for upper and lower indices will be used and the semicolon denotes covariant differentiation using the metric of the undeformed body. The components of the displacement rate are

$$\dot{u}_k = \frac{\partial}{\partial t} u_k(x^m, t).$$

In terms of the symmetric Piola–Kirchhoff stress tensor of the second kind

$$s^{kl} = s^{lk}$$

the equilibrium conditions read ($\delta^k{}_l$ is the unit tensor)

$$[s^{kl}(\delta^m{}_l + u^m{}_{;l})]_{;k} = 0. \tag{3}$$

Here volume and inertia forces have been neglected.

One of the most frequently used multiaxial creep relations is the invariant theory of Odquist[1, 2] which is usually formulated for infinitesimal displacement gradients. Naturally, there is no unique way to generalize this law for large displacement gradients. However, it is possible to conceive two formal procedures for this generalization. Using a spatial reference system one can exchange the infinitesimal strain rate tensor and the deformation rate tensor, and the stress tensor has to be interpreted as Cauchy's stress tensor. Such a relation between the deformation rate tensor and Cauchy's stress tensor definitely satisfies the principle of material objectivity. The material so defined belongs to the class of the incompressible non-Newtonian fluids.

On the other hand if a material reference system is used one may exchange the infinitesimal strain rate tensor by the Lagrange strain rate tensor and the stress may be interpreted by the Piola–Kirchhoff stress tensor of the second kind. Again this relation satisfies the principle of material objectivity. For the following the second approach has been chosen. However, it should be noted that these two generalized constitutive relations do not define the same material if displacement gradients are large. The first relation will satisfy the condition of incompressibility and a hydrostatic pressure will not affect the deformation rate. This does not hold for the second relation. However, if strains but not necessarily the rotations are small, then one can show that the two constitutive relations are equivalent to a first approximation.

The proposed relation reads:

$$\dot{e}_{kl} = f(J_{II})\, {}_0s_{kl}. \tag{4}$$

Here

$${}_0s_{kl} = s_{kl} - \tfrac{1}{3}s^m{}_m g_{kl}$$

is the deviatoric Kirchhoff stress, and g_{kl} are the covariant components of the metric tensor. The term

$$J_{II} = \tfrac{1}{2}{}_0s^k{}_l\, {}_0s^l{}_k$$

is the second invariant of the deviatoric Kirchhoff stress, and f is a scalar-valued function of J_{II}, temperature, and possibly the invariants of the Lagrangian strain tensor if strain hardening is included.

Equation (4) can be transformed to read

$$\dot{e}_{kl} = \frac{\partial W_s}{\partial\, {}_0s^{kl}} \tag{5}$$

where

$$W_s = \int \dot{e}_{kl}\, \mathrm{d}\, {}_0s^{kl} = \int f(J_{II})\, \mathrm{d}J_{II}.$$

According to equation (4)

$$\dot{e}^k{}_k = 0; \tag{6}$$

consequently equation (5) can be written in the form

$$\dot{e}_{kl} = \frac{\partial}{\partial s^{kl}} W_s(s^{ij} - \tfrac{1}{3}s^m{}_m g^{ij}). \tag{7}$$

Conversely we can write

$$ {}_0s^{kl} = \frac{\partial}{\partial \dot{e}_{kl}} W_e(\dot{e}_{ij}), \qquad \dot{e}^k{}_k = 0 \tag{8} $$

where W_e is given by

$$ W_e(\dot{e}_{ij}) = {}_0s^{kl}\dot{e}_{kl} - W_s. \tag{9} $$

From equations (4 and 9) it follows that W_e can be formulated as a function of the second invariant

$$ I_{II} = \tfrac{1}{2}\, {}_0\dot{e}^i{}_j\, {}_0\dot{e}^j{}_i $$

of the deviatoric strain rate tensor

$$ {}_0\dot{e}_{ij} = \dot{e}_{ij} - \tfrac{1}{3}\dot{e}^m{}_m g_{ij}. $$

With equation (6) this yields

$$ {}_0\dot{e}_{ij} = \dot{e}_{ij}. $$

Finally the boundary conditions on the surface $O = O_U + O_T$ will be stated. On part O_U of the surface the velocity field is prescribed and on the remainder O_T the tractions are prescribed. Thus the boundary conditions are

$$ \dot{u}_k = \bar{\dot{u}}_k \qquad \text{on } O_U \tag{10} $$

$$ T^k := s^{lm}(\delta^k{}_m + u^k{}_{;m})n_l = \bar{T}^k \qquad \text{on } O_T. \tag{11} $$

The right-hand sides of equations (10 and 11) are prescribed quantities, and the n_l are the covariant components of the unit normal vector in the undeformed configuration.

The boundary value problem is now defined as follows: Assuming that the displacements are given at some instant t, then the equations (2, 3, 7 or 8), and the boundary conditions (10 and 11) represent a boundary value problem for the displacement rates and stresses. For this problem associated variational principles will be formulated.

3. VARIATIONAL PRINCIPLES

At the instant t the displacement field u_k is assumed to be known. Beside the actual velocity field $\dot{u}_k$ we consider an adjacent one

$$ \dot{u}_k^* = \dot{u}_k + \delta\dot{u}_k $$

which satisfies equation (6) (condition of incompressibility for small strains) and the boundary conditions on O_U equation (10). Thus the virtual velocities $\delta\dot{u}_k$ are subject to the conditions

$$ (\delta\dot{u}^k)_{;k} + (\delta\dot{u}^k)_{;k}\, u_{m;l}\, g^{ml} = 0 \tag{12} $$

$$ \partial\dot{u}_k = 0 \qquad \text{on } O_U. \tag{13} $$

The variation of the dissipation potential W_e is then

$$ \delta W_e = \frac{\partial W_e}{\partial \dot{e}_{kl}}\, \delta\dot{e}_{kl} = {}_0s^{kl}\, \delta\dot{e}_{kl} = s^{kl}\, \delta\dot{e}_{kl} \tag{14} $$

where

$$ \delta\dot{e}_{kl} = \tfrac{1}{2}[(\delta\dot{u}_k)_{;l} + (\delta\dot{u}_l)_{;k} + (\delta\dot{u}^m)_{;k}\, u_{m;l} + u^m{}_{;k}(\delta\dot{u}_m)_{;l}]. $$

The integration of equation (14) over the volume V of the body then yields

$$\int_V \delta W_e \, \mathrm{d}V = \int_V s^{kl} \, \delta \dot{e}_{kl} \, \mathrm{d}V = \int_V s^{kl}(\delta^m{}_l + u^m{}_{;l}) \, \delta \dot{u}_{m;k} \, \mathrm{d}V$$

$$= \int_V [s^{kl}(\delta^m{}_l + u^m{}_{;l}) \, \delta \dot{u}_m]_{;k} \, \mathrm{d}V$$

$$- \int_V [s^{kl}(\delta^m{}_l + u^m{}_{;l})]_{;k} \, \delta \dot{u}_m \, \mathrm{d}V. \tag{15}$$

Using the equilibrium equation (3) and applying the Green–Gauss divergence theorem equation (15) reduces to

$$\delta \int_V W_e \, \mathrm{d}V = \int_O s^{kl}(\delta^m{}_l + u^m{}_{;l}) n_k \, \delta \dot{u}_m \, \mathrm{d}O.$$

With the boundary conditions equations (10 and 11) we obtain

$$\delta \int_V W_e \, \mathrm{d}V - \int_{O_T} \bar{T}^k \, \delta \dot{u}_k \, \mathrm{d}O = 0.$$

If the surface tractions $\bar{T}^k$ are independent of the velocity field the existence of a functional

$$\Phi \colon = \int_V W_e \, \mathrm{d}V - \int_{O_T} \bar{T}^m \dot{u}_m \, \mathrm{d}O \tag{16}$$

can be noted. Φ takes a stationary value for the actual velocity field such that

$$\delta \Phi = 0. \tag{17}$$

This variational principle is subject to the subsidary conditions equations (6 and 10). If we consider the strain rates as new variables, then we get a further subsidiary condition equation (2). Using Lagrange's multiplier method this principle may be transformed such that the three subsidiary conditions become Euler–Lagrange equations. The somewhat lengthy calculation is not given here but can be found in Ref. [10]. In view of the boundary value problem described in Chap. 2 one finds that the multipliers can be interpreted as the stress tensor, the hydrostatic pressure, and the surface tractions on O_U. Consequently the transformed functional reads as follows

$$\Phi_I \colon = \int_V [W_e(\dot{e}_{kl} - \tfrac{1}{3}\dot{e}^m{}_m g_{kl}) - (s^{kl} + p g^{kl}) \dot{e}_{kl}$$

$$+ s^{kl} \tfrac{1}{2}(\dot{u}_{k;l} + \dot{u}_{l;k} + \dot{u}^m{}_{;k} u_{m;l} + u^m{}_{;k} \dot{u}_{m;l})] \, \mathrm{d}V$$

$$- \int_{O_T} \bar{T}^m \dot{u}_m \, \mathrm{d}O - \int_{O_U} T^m(\dot{u}_m - \bar{\dot{u}}_m) \, \mathrm{d}O. \tag{18}$$

This functional will be stationary

$$\delta \Phi_I = 0 \tag{19}$$

for independent variations of $\dot{u}_k$, $\dot{e}_{kl}$, s^{kl}, p and possibly T^m (see equation 11).

From this general theorem further variational theorems are derived. This is done using the general principle that natural conditions may be added as constraints without changing the stationary property of the functional equation (18).

Assuming that the strain rates $\dot{e}_{kl}$ satisfy the approximate equation of incompressibility we express the strain rates by the stresses according to equations (7 and 9). This leads to the functional

$$\Phi_{II}: = \int_V s^{kl}\tfrac{1}{2}(\dot{u}_{k;l} + \dot{u}_{l;k} + \dot{u}^m{}_{;k}u_{m;l} + u^m{}_{;k}\dot{u}_{m;l})$$

$$- W_s(s^{kl} - \tfrac{1}{3}s^m{}_m g^{kl})] \, \mathrm{d}V - \int_{O_T} \bar{T}^m \dot{u}_m \, \mathrm{d}O$$

$$- \int_{O_U} T^m(\dot{u}_m - \hat{u}_m) \, \mathrm{d}O. \tag{19}$$

Here s^{kl}, $\dot{u}_k$ and possibly T^m are to be varied independently to yield the stationary condition

$$\delta\Phi_{II} = 0. \tag{20}$$

It should be noted that the displacement rates do not need to satisfy the approximate incompressibility equation. This principle has a structure similar to the Hellinger–Reissner theorem[4, 6] for an elastic solid or the theorem of Sanders *et al.*[12] for a viscoelastic material.

Another variant can be derived from the functional equation (19) if it is assumed that the stresses do satisfy the equilibrium- and static-boundary conditions equations (3 and 11) respectively. This results in a variational principle for stresses only. However, this variational principle seems to be of limited value and will not be given here.

Finally, expressing the strain rates in equation (18) by the displacement rates (and displacements) we obtain a further variant

$$\Phi_{III}: = \int_V [W_e(\dot{e}_{kl} - \tfrac{1}{3}\dot{e}^m{}_m g_{kl}) - (\dot{u}^k{}_{;k} + \dot{u}^m{}_{;k}u_{m;l}g^{kl})p] \, \mathrm{d}V$$

$$- \int_{O_T} \bar{T}^m \dot{u}_m \, \mathrm{d}O - \int_{O_U} T^m(\dot{u}_m - \hat{u}_m) \, \mathrm{d}O. \tag{21}$$

For the displacement rate principle equation (17) it is possible to derive a sufficient condition for a relative minimum of the functional Φ. Let $\dot{u}_k$ and s^{kl} be the solution of the boundary value problem. Assume that $\delta\dot{u}_k$ satisfies equations (12 and 13).

The Taylor expansion of $\Phi(\dot{u}_k^*)$ yields

$$\Phi(\dot{u}_k^*) = \Phi(\dot{u}_k) + \delta\Phi + \tfrac{1}{2}\,\delta^2\Phi + \cdots$$

which reduces to

$$\Phi(\dot{u}^*{}_k) = \Phi(\dot{u}_k) + \tfrac{1}{2}\,\delta^2\Phi + \cdots;$$

here

$$\delta^2\Phi = \int_V \frac{\partial^2 W_e}{\partial \dot{e}_{ij}\,\partial \dot{e}_{kl}}\,\delta\dot{e}_{ij}\,\delta\dot{e}_{kl}\,\mathrm{d}V.$$

Since W_e is a function of I_{II} we get

$$\frac{\partial^2 W_e}{\partial \dot{e}_{ij}\,\partial \dot{e}_{kl}} = \frac{\partial^2 W_e}{\partial {I_{II}}^2}\,\dot{e}^{ij}\dot{e}^{kl} + \frac{\partial W_e}{\partial I_{II}}\,g^{ik}g^{jl}.$$

Thus

$$\delta^2\Phi = \int_V \left[\frac{\partial^2 W_e}{\partial {I_{II}}^2}(\delta I_{II})^2 + \frac{\partial W_e}{\partial I_{II}}\,\delta\dot{e}^{kl}\,\delta\dot{e}_{kl}\right] \mathrm{d}V$$

where

$$\delta I_{II} = \dot{e}^{ij}\,\delta\dot{e}_{ij}.$$

The sign of $\delta^2\Phi$ is controlled by the sign of the two derivatives of W_e. If

$$\frac{\partial W_e}{\partial I_{II}} > 0 \quad \text{and} \quad \frac{\partial^2 W_e}{\partial {I_{II}}^2} > 0$$

then evidently $\delta^2\Phi$ is positive if only $\delta\dot{e}_{kl} \neq 0$.

Now consider the case

$$\frac{\partial W_e}{\partial I_{II}} > 0 \quad \text{and} \quad \frac{\partial^2 W_e}{\partial {I_{II}}^2} < 0.$$

Using the Schwarz inequality

$$(\delta I_{II})^2 = (\dot{e}^{ij}\,\delta\dot{e}_{ij})^2 \leq (\dot{e}^{ij}\dot{e}_{ij})\,(\delta\dot{e}^{kl}\,\delta\dot{e}_{kl})$$

we get

$$\delta^2\Phi \geq \int_V R\,\delta\dot{e}^{kl}\,\delta\dot{e}_{kl}\,\mathrm{d}V$$

where

$$R = \frac{\partial^2 W_e}{\partial {I_{II}}^2}\,2I_{II} + \frac{\partial W_e}{\partial I_{II}}. \tag{22}$$

Thus $R > 0$ (and $\delta\dot{e}_{kl} \neq 0$) is a sufficient condition for a relative minimum of Φ. The case $\delta\dot{e}_{kl} = 0$ means that the virtual displacement rates produce only a rigid body motion. The case

$$\frac{\partial W_e}{\partial I_{II}} < 0 \quad \text{and} \quad \frac{\partial^2 W_e}{\partial {I_{II}}^2} > 0$$

is of no interest for real materials because the first condition indicates that a specimen under uniaxial tension will reduce its length.

Assume that the constitutive relation reduces to Norton's creep law in the case of infinitesimal deformations and uniaxial stress

$$\dot{e} = K(s)^r$$

where K and r are temperature dependent material constants. Then we get

$$\begin{aligned} f &= \tfrac{3}{2}K(3J_{II})^{(r-1)/2} \\ W_s &= \frac{K}{r+1}(3J_{II})^{(r+1)/2} \\ W_e &= \frac{r}{(r+1)K^{1/r}}(\tfrac{4}{3}I_{II})^{(r+1)/2r}. \end{aligned} \tag{23}$$

Application of the requirement $R > 0$ results in the condition $r > 0$ for the creep exponent. This is generally satisfied by most real materials.

Some remarks will be made concerning the application of the variational principle equation (20). If this principle and the Ritz-method is used to develop an approximate solution, then the dependence of the stress, displacements and consequently displacement rates on the material coordinates have to be suitably represented. After integration of the volume and surface integrals the subsequent variation with respect to the time dependent parameters of the stress-field and the displacement rates (but not displacement parameters) will generally yield a system of nonlinear algebraic equations for these parameters. This is due to the fact that W_s is generally not a quadratic form. However, assume that these equations can be solved for the stress parameters such that the stress parameters can be expressed only by the displacement parameters (but not the time derivatives of the displacement parameters). Consequently the displacement rate parameters, i.e. the time derivatives of the displacement parameters, can be represented as functions of the displacement parameters only. Hence we get a quasilinear system of ordinary differential equation of first order which then has to be integrated with respect to time. For this to be the case the assumed stress and displacement distribution should only linearly depend on their parameters and the number of stress parameters should equal that of the displacement parameters. Unfortunately, these two criteria do not seem to be sufficiently rigorous to assure the reduction of the problem to the solution of a quasilinear differential system of first order. However, the application of the variational theorems equations (17 or 21) will not result in a quasilinear system of ordinary differential equations, since W_e is generally not a quadratic form. It is evident that the solution of such a system needs a considerable numerical effort. Therefore the variational principle equation (20) in some cases may have advantages compared with the theorems equations (17 and 21). For a relatively simple example this will be demonstrated in Chap. 6.

4. KINEMATICS OF THE SHELL DEFORMATION

The theory for a viscous shell to be presented is subject to the following restrictions:

(I) the Kirchhoff–Love hypothesis is assumed to be applicable,
(II) strains are small but rotations are moderately large,
(III) the rotation around the normal to the middle surface is assumed to be small compared to the other two rotations,
(IV) the wall thickness is small compared to the minimum radius of curvature so that in conjunction with the other approximations the metric of the shell space can be approximated by the metric of the middle surface.

The notation will closely follow that of Green and Zerna[13]. Greek indices take the values 1 or 2.

Let $\mathbf{r}$ denote the position vector in the undeformed configuration

$$\mathbf{r} = l[\boldsymbol{\rho}(\Theta^1, \Theta^2) + \lambda\Theta\mathbf{a}_3(\Theta^1, \Theta^2)] \qquad -\tfrac{1}{2} \leq \Theta \leq \tfrac{1}{2}. \tag{24}$$

Here $\boldsymbol{\rho}$ is the dimensionless position vector of the middle surface; $\mathbf{a}_3$ is the unit normal of the middle surface, and Θ is the dimensionless coordinate associated with the unit vector $\mathbf{a}_3$. The quantities l and λ are the minimum radius of curvature and the dimensionless thickness of the shell.

Then the covariant base vectors of the shell space in the reference configuration are given by

$$
\begin{aligned}
\mathbf{g}_\alpha &= l\mu^\gamma_\alpha \mathbf{a}_\gamma, \qquad \alpha = 1, 2 \\
\mathbf{g}_3 &= l\lambda \mathbf{a}_3 \\
\mu^\gamma_\alpha &= \delta^\gamma_\alpha - \lambda\Theta b^\gamma_\alpha, \qquad g = \det(g_{ij}).
\end{aligned} \tag{25}
$$

Here $\mathbf{a}_\alpha$ are the base vectors of the middle surface

$$\mathbf{a}_\alpha = \boldsymbol{\rho}_{,\alpha}.$$

The covariant components of the first and second fundamental form are given by

$$
\begin{aligned}
a_{\alpha\beta} &= \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \\
b_{\alpha\beta} &= a_{\alpha\gamma} b^\gamma_\beta = \mathbf{a}_3 \cdot \mathbf{a}_{\alpha,\beta}.
\end{aligned}
$$

The comma $(\ )_{,\alpha}$ denotes partial differentiation with respects to Θ^α. If $\mathbf{R}$ is the position vector in the deformed configuration then the displacement is given by

$$\mathbf{u} = \mathbf{R} - \mathbf{r}.$$

According to the Kirchhoff–Love hypothesis the displacement vector can be represented by

$$
\begin{aligned}
\mathbf{u} &= l\mathbf{v} + l\lambda\Theta(\mathbf{A}_3 - \mathbf{a}_3) \\
\mathbf{v} &= v^\alpha \mathbf{a}_\alpha + w\mathbf{a}_3.
\end{aligned} \tag{26}
$$

The quantity $\mathbf{A}_3$ denotes the unit normal vector of the deformed middle surface.

If one pictures the coordinate lines $\Theta^\alpha = \text{const.}$ to be convected then the covariant base vectors of the deformed middle surface are given by

$$
\begin{aligned}
\mathbf{A}_\alpha &= \frac{1}{l}(\mathbf{R}_{,\alpha})_{\Theta=0} = \mathbf{a}_\alpha + \mathbf{v}_{,\alpha} \\
&= (\delta^\gamma_\alpha + v^\gamma{}_{;\alpha} - wb^\gamma_\alpha)\mathbf{a}_\gamma - w_\alpha \mathbf{a}_3.
\end{aligned} \tag{27}
$$

The covariant derivative of the surface vector v^γ is defined as

$$v^\gamma{}_{;\alpha} := \frac{\partial v^\gamma}{\partial \Theta^\alpha} + \bar{\Gamma}^\gamma_{\rho\alpha} v^\rho$$

where $\bar{\Gamma}^\gamma_{\rho\alpha}$ is a Christoffel symbol with respect to the surface $\Theta = 0$

$$\bar{\Gamma}^\gamma_{\rho\alpha} = \tfrac{1}{2} a^{\gamma\beta}(a_{\beta\rho,\alpha} + a_{\beta\alpha,\rho} - a_{\rho\alpha,\beta}).$$

The surface tensor w_α is defined as

$$w_\alpha := -(w_{,\alpha} + b_{\alpha\gamma} v^\gamma). \tag{28}$$

The unit normal $\mathbf{A}_3$ of the deformed middle surface is now given by

$$\mathbf{A}_3 = \frac{1}{c}\mathbf{A}_1 \times \mathbf{A}_2 \tag{29}$$

where c is the magnitude of the vector product $\mathbf{A}_1 \times \mathbf{A}_2$.

From the kinematics of shells under infinitesimal displacement gradients it is well known, that the two rotations of the normal of the middle surface are represented by w_α; further, the rigid body rotation of a middle surface element around the normal $\mathbf{a}_3$ is given by

$$\tfrac{1}{2}(v_{1;2} - v_{2;1})\frac{1}{\sqrt{a}}, \qquad a = \det(a_{\alpha\beta})$$

and the infinitesimal strains of the middle surface are defined by

$$\tfrac{1}{2}(v_{\alpha;\beta} + v_{\beta;\alpha} - 2wb_{\alpha\beta}).$$

In view of somewhat similar arguments due to Sanders[14] we will assume that these kinematic quantities are subject to certain order of magnitude restrictions. Without loss of generality we consider the case that the coordinates Θ^α represent the arc length divided by l such that Θ^α are dimensionless. Thus w_α, $v_{\alpha;\beta}$ and $b_{\alpha\beta}$ will be dimensionless. Then the order of magnitude restrictions will be as follows:

$$\begin{aligned} w_\alpha &\sim \lambda \ll 1 \\ \tfrac{1}{2}(v_{\alpha;\beta} - v_{\beta;\alpha}) &\sim (\lambda)^2 \\ \tfrac{1}{2}(v_{\alpha;\beta} + v_{\beta;\alpha} - 2wb_{\alpha\beta}) &\sim (\lambda)^2. \end{aligned} \tag{30}$$

From the last two conditions it follows that

$$v_{\alpha;\beta} - wb_{\alpha\beta} \sim (\lambda)^2.$$

Calculating now the vector product in (29) and dropping all terms of the order of $(\lambda)^2$ or less we get

$$\mathbf{A}_1 \times \mathbf{A}_2 \simeq \sqrt{a}(\mathbf{a}_3 + w^\alpha \mathbf{a}_\alpha).$$

Hence to a first approximation $\mathbf{A}_3$ will become

$$\mathbf{A}_3 \simeq \mathbf{a}_3 + w^\alpha \mathbf{a}_\alpha. \tag{31}$$

Note that $\mathbf{A}_3$ is a unit vector except for an error of the order of $(\lambda)^2$. Combining equations (26 and 31) then gives

$$\mathbf{u} = l[(v^\alpha + \lambda\Theta w^\alpha)\mathbf{a}_\alpha + w\mathbf{a}_3]. \tag{32}$$

This approximate displacement field is equal to that of the engineering shell theories under infinitesimal displacement gradients[13].

The covariant components of the Lagrangian strain tensor $e_{\alpha\beta}$ in the shell space are now given by

$$e_{\alpha\beta} = \tfrac{1}{2}(\mathbf{g}_\alpha \cdot \mathbf{u}_{,\beta} + \mathbf{g}_\beta \cdot \mathbf{u}_{,\alpha} + \mathbf{u}_{,\alpha} \cdot \mathbf{u}_{,\beta}). \tag{33}$$

Equation (33) can be evaluated using equations (25 and 32). Assuming that the covariant derivative $w_{\alpha;\beta}$ is of the order of λ and dropping all terms of the order of $(\lambda)^3$ and less we obtain finally

$$\begin{aligned} e_{\alpha\beta} &= (l)^2(\alpha_{\alpha\beta} + \lambda\Theta\, \omega_{\alpha\beta}) \\ \alpha_{\alpha\beta} &= \tfrac{1}{2}(v_{\alpha;\beta} + v_{\beta;\alpha} - 2wb_{\alpha\beta} + w_\alpha w_\beta) \\ \omega_{\alpha\beta} &= \tfrac{1}{2}(w_{\alpha;\beta} + w_{\beta;\alpha}). \end{aligned} \tag{34}$$

The surface tensor $\alpha_{\alpha\beta}$ is of the order of $(\lambda)^2$; it describes the strain in the middle surface. The surface tensor $\omega_{\alpha\beta}$ is of the order of λ; it defines the curvature changes of the middle surface. Both quantities are not influenced by the rotation around the normal due to assumption equation (30).

It should be noted that the left hand side of equation $(34)_1$ is not a surface tensor, i.e. the indices should be raised or lowered with the contra- or covariant components of the metric tensor

$$g^{\alpha\beta} = \mathbf{g}^{\alpha} \cdot \mathbf{g}^{\beta}$$

$$g_{\alpha\beta} = \mathbf{g}_{\alpha} \cdot \mathbf{g}_{\beta}.$$

However, if only terms of the order of $(\lambda)^2$ are retained in this procedure then this is equivalent to the approximation

$$\mu^{\alpha}_{\beta} \simeq \delta^{\alpha}_{\beta}$$

such that

$$\mathbf{g}_{\alpha} \simeq l\mathbf{a}_{\alpha}$$

$$g_{\alpha\beta} \simeq (l)^2 a_{\alpha\beta} \quad \text{and} \quad g^{\alpha\beta} \simeq \left(\frac{1}{l}\right)^2 a^{\alpha\beta} \tag{35}$$

$$g \simeq (l)^6 (\lambda)^2 a.$$

For the following one should keep in mind that the raising and lowering of indices of the surface tensors $\alpha_{\alpha\beta}$, $\omega_{\alpha\beta}$, v_{α}, w_{α}, etc. is done with the metric $a^{\alpha\beta}$ and $a_{\alpha\beta}$; but that of the strain and stress tensors $e_{\alpha\beta}$ and $s^{\alpha\beta}$ is done with the metric equation (35). Within the above approximations the factors $(l)^2$ and $(^1/l)^2$ are only due to the introduction of a dimensionless position vector $\boldsymbol{\rho}$ of the reference middle surface.

5. A VARIATIONAL PRINCIPLE FOR A THIN VISCOUS SHELL

The basis for the following derivations will be the mixed variational principle equation (20). The displacements in the shell space and the strains will be given by equations (32 and 34) respectively. In view of the Kirchhoff–Love hypothesis the shear stresses and the direct stress normal to the middle surface will not be considered in the constitutive relations. This means that terms which contain these stress components will be dropped in the volume integral of the functional Φ_{II} equation (19). Assuming that

$$\delta\dot{\mathbf{u}} = 0 \quad \text{on} \quad O_U$$

then equation (19) reads

$$\Phi_{II} = \int_v [s^{\alpha\beta}\dot{e}_{\alpha\beta} - W_s(s^{\alpha\beta} - \tfrac{1}{3}s^{\rho}_{\rho}g^{\alpha\beta})]\,\mathrm{d}V - \int_{O_T} \bar{\mathbf{T}} \cdot \dot{\mathbf{u}}\,\mathrm{d}O$$

where

$$\mathrm{d}V = \sqrt{g}\,\mathrm{d}\Theta^1\,\mathrm{d}\Theta^2\,\mathrm{d}\Theta \simeq \lambda l\,\mathrm{d}\Theta\,\mathrm{d}f.$$

The surface element $\mathrm{d}f$ of the middle surface is given by

$$\mathrm{d}f = (l)^2\sqrt{a}\,\mathrm{d}\Theta^1\,\mathrm{d}\Theta^2.$$

Using equation (34) we get

$$\Phi_{II} = \int_f (n^{\alpha\beta}\dot{\alpha}_{\alpha\beta} + m^{\alpha\beta}\dot{\omega}_{\alpha\beta} - W_s^*)\,\mathrm{d}f - \int_{O_T} \bar{\mathbf{T}}\cdot\dot{\mathbf{u}}\,\mathrm{d}O.$$

Here we have introduced the definitions

$$\begin{aligned} n^{\alpha\beta} &:= \lambda(l)^3 \int_{-1/2}^{+1/2} s^{\alpha\beta}\,\mathrm{d}\Theta = a^{\alpha\gamma}n_\gamma^\beta \\ m^{\alpha\beta} &:= (\lambda)^2(l)^3 \int_{-1/2}^{+1/2} s^{\alpha\beta}\Theta\,\mathrm{d}\Theta = a^{\alpha\gamma}m_\gamma^\beta \\ W_s^* &:= \lambda l \int_{-1/2}^{+1/2} W_s(s^{\alpha\beta} - \tfrac{1}{3}s^\rho{}_\rho g^{\alpha\beta})\,\mathrm{d}\Theta. \end{aligned} \tag{36}$$

The rate tensors $\dot{\alpha}_{\alpha\beta}$ and $\dot{\omega}_{\alpha\beta}$ are given by

$$\begin{aligned} \dot{\alpha}_{\alpha\beta} &= \tfrac{1}{2}(\dot{v}_{\alpha;\beta} + \dot{v}_{\beta;\alpha} - 2\dot{w}b_{\alpha\beta} + \dot{w}_\alpha w_\beta + w_\alpha \dot{w}_\beta) \\ \dot{\omega}_{\alpha\beta} &= \tfrac{1}{2}(\dot{w}_{\alpha;\beta} + \dot{w}_{\beta;\alpha}) \\ \dot{w}_\alpha &= -(\dot{w}_{,\alpha} - b_{\alpha\gamma}\dot{v}^\gamma). \end{aligned} \tag{37}$$

The surface integral on O_T is approximately equal to an integral on the middle surface and integrals along the edge C. It is assumed that the loading of the middle surface is due to a hydrostatic pressure p.

Then

$$\bar{\mathbf{T}} = -p\mathbf{A}_3\frac{\mathrm{d}F}{\mathrm{d}f}$$

where $\mathrm{d}F$ is the element of the deformed middle surface. For small strains we may approximate this using equation (31) by

$$\bar{\mathbf{T}} \simeq -p(\mathbf{a}_3 + w^\alpha\mathbf{a}_3).$$

A more accurate analysis is given in Ref. [10]. If we define the edge loading to be

$$\begin{aligned} \bar{n}^\alpha &:= \lambda l \int_{-1/2}^{1/2} \bar{\mathbf{T}}\cdot\mathbf{a}_\alpha\,\mathrm{d}\Theta \\ \bar{m}^\alpha &:= (\lambda)^2 l \int_{-1/2}^{1/2} \bar{\mathbf{T}}\cdot\mathbf{a}_\alpha\,\Theta\,\mathrm{d}\Theta \\ \bar{q} &:= (\lambda)^2 l \int_{-1/2}^{1/2} \bar{\mathbf{T}}\cdot\mathbf{a}_3\,\mathrm{d}\Theta \end{aligned}$$

then the surface integral is given by

$$\Pi = -\int_f pl(\dot{w} + w_\alpha\dot{v}^\alpha)\,\mathrm{d}f + l\int_C (\bar{n}^\alpha\dot{v}_\alpha + \bar{m}^\alpha\dot{w}_\alpha + \bar{q}\dot{w})\,\mathrm{d}C$$

where C denotes the arc length along the edge of the reference middle surface. The term $\bar{m}^\alpha\dot{w}_\alpha$ can be partially integrated using

$$\dot{w}_\alpha = -l(v_\alpha\dot{w}_{,v} - \varepsilon_{\alpha\beta}v^\beta\dot{w}_{,C}) - b_\alpha^\gamma\dot{v}_\gamma.$$

Here $\varepsilon_{\alpha\beta}$ is the permutation tensor and ν denotes the arc length along the outer unit normal vector

$$\mathbf{v} = \nu^\alpha \mathbf{a}_\alpha$$

of the edge C. The final result for the functional Φ_{II} is

$$\Phi_{II} = \int_f [n^{\alpha\beta}\dot{\alpha}_{\alpha\beta} + m^{\alpha\beta}\dot{\omega}_{\alpha\beta} - W_s^* + pl(\dot{w} + w_{,\rho}\dot{v}^\rho)]\,\mathrm{d}f$$

$$- l\int_C [(\bar{n}^\alpha - \bar{m}^\gamma b_\gamma^\alpha)\dot{v}_\alpha - (\bar{m}^\alpha \nu_\alpha)\dot{w}_{,\nu}\, l + \bar{q}^*\dot{w}]\,\mathrm{d}C \tag{38}$$

where $\bar{q}^*$ is the ersatz-shear force at the edge

$$\bar{q}^* = \bar{q} - l(\bar{m}^\alpha \varepsilon_{\alpha\beta}\, \nu^\beta)_{,C}.$$

Generally for most metals the stress distribution across the shell is nonlinear. Thus if the stresses are of primary interest and the principle equation (20) is used to obtain an approximate solution then a series development of the stresses in Θ or a multi-zone concept has to be used. However if deformations are primarily considered then a linear variation of the stresses across the shell can be assumed, or the double-membrane concept may be used to give sufficiently accurate results. In these cases the stresses $s^{\alpha\beta}$ are completely determined by the moments and membrane forces such that the function W_s can be expressed by the stress resultants. The further discussion will be restricted to these cases. The independent variation of the displacement rates $\dot{v}_\alpha$, $\dot{w}$ and the stress resultants $n^{\alpha\beta}$, $m^{\alpha\beta}$ and the use of the Green–Gauss theorem then results in the following necessary conditions for the stationarity of the functional equation (38):

constitutive relations:

$$\dot{\alpha}_{\alpha\beta} = \frac{\partial W_s^*}{\partial n^{\alpha\beta}}, \qquad \dot{\omega}_{\alpha\beta} = \frac{\partial W_s^*}{\partial m^{\alpha\beta}}$$

equilibrium conditions:

$$n^{\beta\alpha}{}_{;\beta} + n^{\rho\beta} w_{,\rho}\, b_\beta^\alpha - m^{\rho\beta}{}_{;\rho}\, b_\beta^\alpha - plw^\alpha = 0$$

$$n^{\alpha\beta} b_{\alpha\beta} - (n^{\alpha\beta} w_{,\alpha})_{;\beta} + m^{\alpha\beta}{}_{;\alpha\beta} - pl = 0$$

boundary conditions:

$$\bar{n}^\alpha - \bar{m}^\gamma b_\gamma^\alpha - n^{\alpha\beta}\nu_\beta + m^{\rho\beta} b_\beta^\alpha \nu_\rho = 0$$

or $\dot{v}_\alpha$ prescribed,

$$\bar{m}^\alpha \nu_\alpha - m^{\alpha\beta}\nu_\alpha \nu_\beta = 0$$

or $\dot{w}_{,\nu}$ prescribed,

$$\bar{q}^* + n^{\alpha\beta} w_{,\alpha} \nu_\beta - m^{\alpha\beta}{}_{;\alpha}\nu_\beta - l(m^{\alpha\rho}\nu_\rho \varepsilon_{\alpha\beta}\, \nu^\beta)_{,C} = 0$$

or $\dot{w}$ prescribed.

The equilibrium and boundary conditions and the strain–displacement relations are equivalent to those given by Sanders[14] (equations 89–96) if shear forces are eliminated.

6. APPLICATION TO THE CREEP COLLAPSE OF A CYLINDRICAL SHELL UNDER UNIFORM HYDROSTATIC PRESSURE

The problem is characterized by the fact that initial deviations of the perfect circular geometry (e.g. ovality) will induce circumferential bending moments which then produce a progressive increase of the deviations (ovality) due to nonuniform creep. We will assume that the geometry and loading does not change along the length of the shell and that the middle surface is given by (Fig. 1):

$$\boldsymbol{\rho} = (1 + \alpha \cos 2\phi)(\cos \phi \mathbf{e}_1 + \sin \phi \mathbf{e}_3) + x\mathbf{e}_2 . \tag{39}$$

The term $\alpha \cos 2\phi$ indicates that the cross-section is assumed to be quasi-elliptic where $\alpha \ll 1$ is a measure of the ovality.

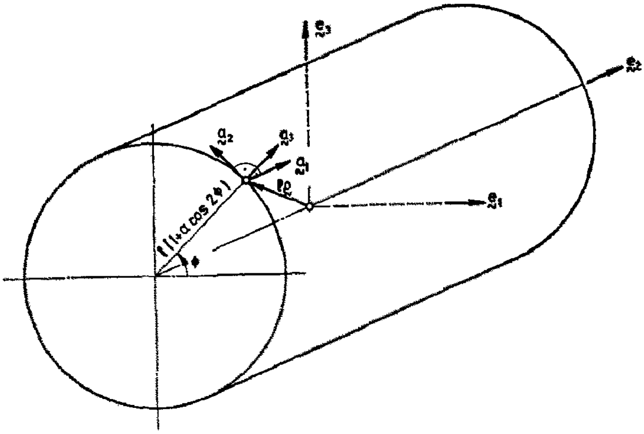

Fig. 1. Coordinate system of the quasi-elliptic cylinder.

In the following we will formulate all equations using physical components of the tensors; the relevant physical components are given by

$$u = v^1, \qquad v = v^2\sqrt{a_{22}}$$

$$n_x = n_1^1, \qquad n_\phi = n_2^2$$

$$m_x = \frac{1}{l} m_1^1, \qquad m_\phi = \frac{1}{l} m_2^2$$

where l denotes the average radius of the shell.

From equation (39) the relevant geometric and kinematic quantities can be calculated using a series development with respect to α. If only terms linear in α are retained then the physical components of the strain and the change of curvature in the circumferential direction are given by

$$\begin{aligned} \alpha_\phi = \alpha_2^2 &= v_{,\phi}(1 - \alpha \cos 2\phi) + w(1 + 3\alpha \cos 2\phi) \\ &+ \tfrac{1}{2}(v)^2(1 + 6\alpha \cos 2\phi) \\ &- w_{,\phi} v(1 + 2\alpha \cos 2\phi) \\ &+ \tfrac{1}{2}(w_{,\phi})^2(1 - 2\alpha \cos 2\phi) \end{aligned} \tag{40}$$

$$\omega_\phi = \omega_2^2 = -[w_{,\phi\phi}(1 - 2\alpha\cos 2\phi) + w_{,\phi}2\alpha\sin 2\phi$$
$$- v_{,\phi}(1 + 2\alpha\cos 2\phi) + v6\alpha\sin 2\phi].$$

Further the element of the reference middle surface for unit axial length is given by

$$\mathrm{d}f = l\sqrt{a}\,\mathrm{d}\phi \simeq l(1 + \alpha\cos 2\phi)\,\mathrm{d}\phi.$$

For the constitutive equation it will be assumed that equation (4) reduces to Norton's creep law in the case of infinitesimal deformation and uniaxial stresses. The multiaxial constitutive relation is defined by equation (23).

Assuming that the axial creep rate $\dot{e}_1^1$ vanishes it follows from equation (4) that

$$s_1^1 = \tfrac{1}{2}s_2^2$$

such that

$$J_{II} = \tfrac{1}{4}(s_2^2)^2$$
$$W_s = \frac{k}{r+1}\,|s_2^2|^{r+1}$$
$$k = K(\tfrac{3}{4})^{(r+1)/2}.$$

For reasons of simplicity we will use the well known double membrane concept (Fig. 2).

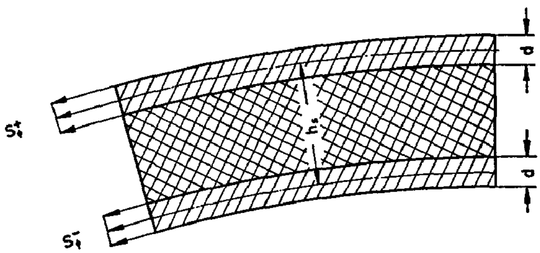

Fig. 2. Geometry of the double membrane.

The physical components of the circumferential stresses in the inner and outer membrane are then given by

$$s_\phi = s_2^2 = \begin{Bmatrix} s_\phi^{\,+} \\ \\ s_\phi^{\,-} \end{Bmatrix} = \begin{Bmatrix} n_\phi + m_\phi \dfrac{1}{h_s/2} \\ \\ n_\phi - m_\phi \dfrac{1}{h_s/2} \end{Bmatrix} \frac{1}{2d}.$$

Hence it follows

$$W_s^* = h\int_{-1/2}^{1/2} W_s\,\mathrm{d}\Theta = \frac{k}{r+1}\,\frac{d}{(2d)^{r+1}}\left[\left|n_\phi + \frac{m_\phi}{h_s/2}\right|^{r+1} + \left|n_\phi - \frac{m_\phi}{h_s/2}\right|^{r+1}\right]. \qquad (41)$$

The functional equation (38) may then be represented by

$$\Phi_{II} = \int_0^{2\pi}\left\{n_\phi\,\dot{\alpha}_\phi + \frac{1}{l}\,m_\phi\,\dot{\omega}_\phi - W_s^* + pl[\dot{w} + \dot{v}v(1 + 3\alpha\cos 2\phi)\right.$$
$$\left. - \dot{v}w_{,\phi}(1 - \alpha\cos 2\phi)]\right\} l(1 + \alpha\cos 2\phi)\,\mathrm{d}\phi \qquad (42)$$

Considering the symmetry properties of the shell the first terms of a series development of the relevant stress resultants and displacements are

$$\begin{aligned} n_\phi &= n_I(t) + n_{II}(t)\cos 2\phi \\ m_\phi &= \qquad\quad m_{II}(t)\cos 2\phi \\ w &= w_I(t) + w_{II}(t)\cos 2\phi \\ v &= \qquad\quad v_{II}(t)\sin 2\phi \end{aligned} \tag{43}$$

where n_I, n_{II}, m_{II} and w_I, w_{II}, v_{II} are the time dependent parameters. It should be noted that the bending moment m_ϕ does not depend on a constant term. This is suggested by the corresponding solution of a linear viscous shell or ring. As a consequence the number of displacement parameters equals that of the stress resultants. This approximation may be further simplified if one neglects the variation of membrane force along the circumference; thus

$$n_{II} \simeq 0.$$

As a consequence of this the strains of the middle surface should not vary in the circumferential direction. To a first approximation we thus find from equation (40)

$$v_{II} \simeq -\tfrac{1}{2} w_{II}.$$

With these simplifications again the number of the displacement parameters equals that of the stress resultants. The integration of the functional equation (42) then gives

$$\Phi_{II} = \pi l \left\{ n_I[2\dot{w}_I + \dot{w}_{II}(4\alpha + \tfrac{9}{4} w_{II})] + m_{II}\frac{3}{l}\dot{w}_{II} - W_s^{**} + pl[2\dot{w}_I + \dot{w}_{II}(\alpha - \tfrac{3}{4} w_{II})] \right\}. \tag{44}$$

Here

$$\begin{aligned} W_s^{**} &= \int_0^{2\pi} W_s^*(1 + \alpha\cos 2\phi)\, \mathrm{d}\phi \\ &= \frac{k}{r+1}(2d)^{-r}(\operatorname{sgn} n_I)^{r+1} \sum_{\nu=0,2,4\ldots}^{\bar{r}} \binom{r+1}{\nu} A_\nu (n_I)^{r+1-\nu} \left(\frac{m_{II}}{h_s/2}\right)^{\nu} \end{aligned}$$

where

$$\begin{aligned} A_\nu &= \frac{1}{\pi}\int_0^{2\pi} (\cos 2\phi)^\nu\, \mathrm{d}\phi, && \nu = 0, 1, 2, \ldots, r+1 \\ A_0 &= 2 \\ A_\nu &= A_{\nu-2}\frac{\nu - 1}{2}, && \nu = 2, 4, \ldots \\ A_\nu &= 0, && \nu = 1, 3, \ldots \end{aligned}$$

This result is restricted to integer creep exponents such that

$$\bar{r} = \begin{Bmatrix} r+1, & r \text{ odd} \\ r, & r \text{ even} \end{Bmatrix}.$$

If the creep exponent r is an even number then this result is further restricted by the condition

$$\left|\frac{m_{II}}{n_I h_s/2}\right| \leq 1.$$

The independent variation of n_I, m_{II}, $\dot{w}_I$ and $\dot{w}_{II}$ then yield four necessary equations for the stationary condition of Φ_{II}:

$$\frac{\partial \Phi_{II}}{\partial n_I} = 0, \qquad \frac{\partial \Phi_{II}}{\partial m_{II}} = 0$$

$$\frac{\partial \Phi_{II}}{\partial \dot{w}_I} = 0, \qquad \frac{\partial \Phi_{II}}{\partial \dot{w}_{II}} = 0.$$

The first two equations represent the averaged constitutive relations and the last represent the averaged equation of equilibrium. The first equation is a quasilinear differential equation for the function w_I and will not be given here. The others are

$$\frac{3}{l}\dot{w}_{II} - \frac{k2d}{(r+1)h_s/2}\left[\frac{\operatorname{sgn} n_I}{2d}\right]^{r+1} \sum_{\nu=2,4,\ldots}^{\bar{r}} \binom{r+1}{\nu} A_\nu \nu (n_I)^{r+1-\nu}\left(\frac{m_{II}}{h_s/2}\right)^{\nu-1} = 0$$

$$2n_I + 2pl = 0 \tag{45}$$

$$n_I(4\alpha + \tfrac{9}{4}w_{II}) + m_{II}\frac{3}{l} + pl(\alpha - \tfrac{3}{4}w_{II}) = 0.$$

Thus

$$n_I = -pl, \qquad m_{II} = p(l)^2(\alpha + w_{II})$$

and defining

$$\mu := \nu - 1$$

$$\chi := \frac{2m_{II}}{plh_s} = (\alpha + w_{II})\frac{2l}{h_s}$$

$$r^* := \begin{Bmatrix} r, & r \text{ odd} \\ r-1, & r \text{ even} \end{Bmatrix}$$

we get from equation $(45)_1$ with $p > 0$

$$\dot{\chi} = \frac{4}{3}\left(\frac{l}{h_s}\right)^2 k \left(\frac{pl}{2d}\right)^r \sum_{\mu=1,3,\ldots}^{r^*} \binom{r}{\mu} A_{\mu+1}(\chi)^\mu. \tag{46}$$

The initial condition is

$$w_{II} = 0 \quad \text{or} \quad \chi = \alpha\frac{2l}{h_s}.$$

Hence the result is a quasilinear differential equation of first order which describes the change of the ovality. It is interesting to note that the application of the theorem equation (17) would have led to a system of nonlinear equations for $\dot{w}_I$ and $\dot{w}_{II}$. For the problem considered this clearly shows the advantage of the mixed variational theorem.

The character of the solution of equation (46) is well known[9]. For exponents $r > 1$ a critical time t_c exists where the deformation χ theoretically increases beyond limits. For $r = 1$ the deformation increases exponentially but there exists no asymptotic time t_c.

A comparison with the theory of Hoff *et al.*[9] shows that the main difference between the two theories lies in the terms $A_{\mu+1}$. If $(A_{\mu+1})_H$ denotes the value derived by Hoff *et al.* then the ratios for $\mu = 1, 3, 5, 7, 9$ are as follows:

$$\frac{A_{\mu+1}}{(A_{\mu+1})_H} = 1{\cdot}0,\ 1{\cdot}127,\ 1{\cdot}172,\ 1{\cdot}193,\ 1{\cdot}211.$$

This indicates that the rate of deformation is somewhat larger in the theory presented here. The difference is due to the fact that the weighting factor in the variational averaging procedure is different from that of the continuous collocation method of Hoff *et al.* However, in view of the uncertainties of the creep parameters this difference is not significant. Although the approach of Hoff *et al.* is much simpler from the mathematical standpoint the author thinks that the mechanical interpretation of the method of Hoff *et al.* is somewhat difficult because an explicit distinction between spatial and material coordinates is not made. On the other hand the continuous collocation method used in Ref. [9] does not lend itself to further improved solutions; it is applicable to approximate deformation processes with only one degree of freedom.

REFERENCES

1. F. K. G. Odqvist and J. Hult, *Kriechfestigkeit metallischer Werkstoffe.* Springer (1962).
2. F. K. G. Odqvist, *Mathematical Theory of Creep and Creep Rupture.* Clarendon Press, Oxford (1966).
3. Y. N. Rabotnov, *Creep Problems in Structural Members.* North-Holland, Amsterdam–London (1969).
4. E. Hellinger, Die allgemeinen Ansätze der Mechanik der Kontinua, *Enz. math. Wiss.* **4**, 602 (1914).
5. E. Reissner, On a variational theorem in elasticity, *J. Math. Phys.* **27**, 90 (1950).
6. E. Reissner, On a variational theorem for finite elastic deformations, *J. Math. Phys.* **32** (2/3), 129 (1953).
7. Y. N. Rabotnov, Axisymmetrical Creep Problems of circular cylindrical shells, *J. app. Math. Mech.* **28**, 1255 (1964).
8. E. I. Grigoliuk and Y. V. Lipovtsev, Use of the variational principle in stability problems of shells under creep conditions, *Mechanics of Solids* **1**, 56 (1966).
9. N. J. Hoff, W. E. Jahsman and W. Nachbar, A study of creep collapse of a long circular cylindrical shell under uniform external pressure, *J. Aerospace Sci.* 663 (Oct. 1959).
10. T. Malmberg, Über einige Variationsprinzipe der Kriechmechanik und ihre Anwendung auf das Kriechkollaps-Problem von Brennstabhüllrohren, Thesis, Techn. Univ. Karlsruhe (1971); also report *KFK* 1390 of the Nuclear Research Center Karlsruhe, Germany (1971).
11. T. Malmberg, On Some Variational Principles of Creep Mechanics with Applications to the Problem of Creep Collapse of Fuel Element Claddings. Paper C 2/4, *Proc. 1st Int. Conf. on Struct. Mech. in Reactor Techn.*, Berlin (20–24 Sept. 1971).
12. J. L. Sanders, H. G. McComb and F. R. Schlechte, A Variational Theorem for Creep Applications to Plates and Columns, *NACA TN* 4003 (1957).
13. A. E. Green and W. Zerna, *Theoretical Elasticity.* Clarendon Press, Oxford (1960).
14. J. L. Sanders, Nonlinear theories for thin shells, *Quart. appl. Math.* **21**, 21 (1963).

Абстракт — Постулируется конститутивная зависимость для вязкого материала, подверженного действию малых деформаций, но конечннх вращений. Даются формулы. для связанных вариационных теорем. Они подобны к принципу минимума потенциальной энергии и теоремы Геллингера — Рейсснера для упругого твердого тела. Даются выводы зависимостей деформация — перемещение для тонких оболочек, подверженных действию малых деформаций, но умеренно больших вращений. На этой основе определяется смешанный вариационный принцип для тонких вязных оболочек. Указывается для задачи разрушения вследствие ползучести длинных цилиндрических оболочек, подверженных действию внешнего давления, что смешанный вариационный принцип можно сравнить, как присоединенный к другим вариационным теоремам. Дается, также, сравнение с теорией разрушения вследствие ползучести, предложенной Хоффом, Джасменом и Нахбаром.